Lecture 24

Thursday, 30 April 2026 9:24 AM

Bimatrix Games.

Example: The Canteen Game (E) (W)

	E	W
E	4 \ 7	0 \ 0
W	3 \ 3	7 \ 4

R =

	E	W
E	4	0
W	3	7

C =

	E	W
E	7	0
W	3	4

R & C are called the payoff matrices for the row player (R) & the column player C respectively. Say $R, C \in \mathbb{R}^{n \times m}$.
Then R has n pure strategies, C has m pure strategies.
(i,j) where $i \in [n]$, $j \in [m]$ is called a pure strategy profile.

Then for (i,j), R's utility is $R_{ij} = e_i^T R e_j$
C's utility is $C_{ij} = e_i^T C e_j$

We say i is a best response to j if $R_{ij} \geq R_{i'j}$ $\forall i' \in [n]$
Define $BR_R(j) = \arg_{i \in [n]} R_{ij}$.

Then (i,j) is a PNE if:
$R_{ij} \geq R_{i'j}$ $\forall i' \in [n]$ (or $i \in BR_R(j)$)
$C_{ij} \geq C_{ij'}$ $\forall j' \in [m]$ (or $j \in BR_C(i)$)

Example: RPS $R > S > P$

Mixed strategies, strategy profile, probabilities, expected utilities.
Δ_n, Δ_m, x, y

Support

Best response

MNE, characterization
(x,y) is an MNE if:
$\forall x' \in \Delta_n$, $x^T R y \geq x'^T R y$
$\forall y' \in \Delta_m$, $x^T C y \geq x^T C y'$

Nash's Theorem

Characterization in terms of support

LP for MNE, given supports.

Zero-sum games

Finding equilibria in zero-sum games:

Let (R, C) be a zero-sum game. Thus, $C = -R$.
Consider the column player's perspective. Suppose it plays x. Then the row player's utilities for it's strategies are $x^T C$ (this is a row vector).
If the column player chooses best-response to x, it gets $\max_j (x^T C)_j$, and hence the row player gets $-\max_j (x^T C)_j = \min_j (x^T R)_j$

Since at equilibrium both players best-respond to each other (by defn.), the row player "should" choose x to maximize it's utility when column player best-responds, i.e., choose x to maximize $\min_j (x^T R)_j$

Note that we are not saying that such an x is an equilibrium strategy, in particular why x is a best-response to the column player's strategy (it may not be!)

Example.

R =

5	-5
-10	10

If we restrict to pure strategies, R should play 1.
for mixed strategies, R should play $(2/3, 1/3)$

We can find such an x by an LP: $\boxed{P_R}$

$$\begin{aligned} \max \quad & w \\ \text{s.t.} \quad \forall j \quad (x^T R)_j &\geq w \\ \sum_i x_i &= 1 \\ x &\geq 0 \end{aligned}$$

Similarly, for the column player, a good strategy would be to choose y which optimizes:

$\boxed{P_C}$

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad \forall i \quad (Cy)_i &\geq z \\ \sum_j y_j &= 1 \\ y &\geq 0 \end{aligned} \qquad \text{OR} \qquad \begin{aligned} \max \quad & z \\ \forall i \quad z - (Cy)_i &\leq 0 \quad \times x_i \\ \sum_j y_j &= 1 \quad \times w' \\ y &\geq 0 \end{aligned}$$

Let us write the dual of P_C. This is:

$$\begin{aligned} \min \quad & w' \\ \forall j \quad -(x^T C)_j + w' &\geq 0 \\ \sum_i x_i &= 1 \\ x &\geq 0 \end{aligned} \qquad \equiv \qquad \begin{aligned} -\max \quad & w \\ \forall j \quad (x^T R)_j &\geq w \\ \sum_i x_i &= 1 \\ x &\geq 0 \end{aligned} \qquad \boxed{D_C}$$

Note that D_C is nearly the same as P_R, except that the objective value gets negated. I.e., (x^*, y^*) is optimal for D_R iff (x^*, w^*) is optimal for D_C

Let (x^*, w^*) be optimal for P_R, & (y^*, z^*) be optimal for P_C. Then by strong duality, $-z^* = w^*$.

Theorem: (x^*, y^*) is a NE

Proof: We need to show that for the row player, x^* is a best-response to y^*, i.e., $\forall x$, $x^T R y^* \leq x^{*T} R y^*$.

Consider y^*. We know that if column player plays y^*, and if row-player best-responds, column player gets z^* (negation of D_R). Thus, row-player gets $-z^*$. Thus for any response to y^*, row-player gets at most $-z^*$.
$\forall x \quad x^T R y^* \leq -z^* = w^*$
Now consider x^*, Similar to above, for any strategy y, row player gets at least w^*.
$\forall y \quad x^{*T} R y^* \geq w^*$.
Thus, $x^{*T} R y^* \geq x^T R y^*$ $\forall x$, and hence x^* is a best response to y^*.
Similarly we can show that y^* is a best-response to x^*, and hence (x^*, y^*) is a NE ∎

Note:
(i) The proof holds for any optimal soln. x^* to P_R, and any optimal soln. y^* to P_C.
(ii) For any such x^*, y^*, the row-player's utility at equilibrium is w^*. Hence, there are multiple equilibria, but the row-player's payoff (and hence, the column player's payoff) is exactly the same.
The value w^* is called the value of the zero-sum game
(iii) At equilibrium, each player is playing a min-max strategy, or a risk-averse strategy.
In general games, a min-max strategy does not give an equilibrium.